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# INVARIANT SOLUTIONS OF RANK 1 OF THE EQUATIONS OF PLANE MOTION OF A VISCOUS HEAT-CONDUCTING PERFECT GAS

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A system of equations of plane motion of a viscous heat-conducting perfect gas is considered:

$$\rho(u_t + uu_x + vu_y) = -p_x + \frac{2}{3}(\mu(2u_x - v_y))_x + (\mu(u_y + v_x))_y; \quad (1)$$

$$\rho(v_t + uv_x + vv_y) = -p_y + (\mu(u_y + v_x))_x + \frac{2}{3}(\mu(2v_y - u_x))_y; \quad (2)$$

$$\rho_t + (u\rho)_x + (v\rho)_y = 0; \quad (3)$$

$$p_t + up_x + vp_y + \gamma p(u_x + v_y)$$
$$= \frac{\gamma - 1}{R} k_0\Big(\Big(\mu\Big(\frac{p}{\rho}\Big)_x\Big)_x + \Big(\mu\Big(\frac{p}{\rho}\Big)_y\Big)_y\Big) + (\gamma - 1)\mu\Big(\frac{4}{3}(u_x^2 + v_y^2 - u_xv_y) + (v_x + u_y)^2\Big). \quad (4)$$

Here $u$ and $v$ are the coordinates of the velocity vector, $\rho$ is the density, $p$ is the pressure, $\mu = (p/\rho)^\omega$ is the viscosity coefficient, $k_0\mu$ is the heat-conductivity coefficient, $\gamma$ is the ratio of specific heats, and $R$ is Boltzmann's constant.

The goal of this work is to construct all invariant solutions of rank 1 for system (1)–(4) [1].

As is shown in [2], system (1)–(4) admits the Lie algebra $L_8$ with the basis

$$X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = t\partial_x + \partial_u, \qquad X_4 = t\partial_y + \partial_v,$$
$$X_5 = y\partial_x - x\partial_y + v\partial_u - u\partial_v, \qquad X_6 = \partial_t,$$
$$X_7 = t\partial_t + x\partial_x + y\partial_y - \rho\partial_\rho - p\partial_p,$$
$$X_8 = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2(\omega - 1)\rho\partial_\rho + 2\omega p\partial_p.$$

Table 1 is the commutator table for the operators of the Lie algebra $L_8$. Here and below, admissible operators are represented by their numbers.

Table 2 gives actions of the internal automorphisms of the Lie algebra $L_8$ on the coordinates of the vector $X = x^iX_i$. To these we add the discrete automorphism $E_1$, which corresponds to the reversal of the direction of the $x$ axis. The actions of the automorphisms $A_1$ and $A_2$ are combined into the action of the automorphism $T$, and $A_3$ and $A_4$ are combined into the action of the automorphism $\Gamma$. The group parameter of the automorphism $A_i$ $(i = 1, \ldots, 8)$ is denoted by $a_i$. Also, the following notation is used: $\alpha_1 = (a_1, a_2)$, $\alpha_2 = (a_3, a_4)$, $\alpha_7 = \exp\{a_7\}$, and $\alpha_8 = \exp\{a_8\}$,

$$S = \begin{pmatrix} \cos a_5 & \sin a_5 \\ -\sin a_5 & \cos a_5 \end{pmatrix}, \qquad \Delta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Table 3 gives the normalized optimal system of subalgebras of the Lie algebra $L_8$ [3]. The first integer in the subalgebra number denotes its dimension, and the second denotes its number among the subalgebras of the given dimension, $N$ is the basis of the subalgebra, and Nor$N$ is its normalizer. The coefficients $\alpha$ and $\beta$ take any real value. The coefficients $\delta$ and $\varepsilon$ do not take values of 0 and $-1$, respectively. The coefficient $\zeta$

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TABLE 1

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | −2 | 0 | 1 | 1 |
| 2 | 0 | 0 | 0 | 0 | 1 | 0 | 2 | 2 |
| 3 | 0 | 0 | 0 | 0 | −4 | −1 | 0 | 3 |
| 4 | 0 | 0 | 0 | 0 | 3 | −2 | 0 | 4 |
| 5 | 2 | −1 | 4 | −3 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 1 | 2 | 0 | 0 | 6 | 0 |
| 7 | −1 | −2 | 0 | 0 | 0 | −6 | 0 | 0 |
| 8 | −1 | −2 | −3 | −4 | 0 | 0 | 0 | 0 |

TABLE 2

| Automorphism | $p_1 = (x^1, x^2)$ | $p_2 = (x^3, x^4)$ | $x^5$ | $x^6$ | $x^7$ | $x^8$ |
|---|---|---|---|---|---|---|
| $T$ | $p_1 + \alpha_1 \Delta x^5 + \alpha_1(x^7, x^8)$ | $p_2$ | $x^5$ | $x^6$ | $x^7$ | $x^8$ |
| $\Gamma$ | $p_1 - \alpha_2 x^6$ | $p_2 + \alpha_2 \Delta x^5 + \alpha_2 x^8$ | $x^5$ | $x^6$ | $x^7$ | $x^8$ |
| $A_5$ | $p_1 S$ | $p_2 S$ | $x^5$ | $x^6$ | $x^7$ | $x^8$ |
| $A_6$ | $p_1 + a_6 p_2$ | $p_2$ | $x^5$ | $x^6 + a_6 x^7$ | $x^7$ | $x^8$ |
| $A_7$ | $\alpha_7 p_1$ | $p_2$ | $x^5$ | $\alpha_7 x^6$ | $x^7$ | $x^8$ |
| $A_8$ | $\alpha_8 p_1$ | $\alpha_8 p_2$ | $x^5$ | $x^6$ | $x^7$ | $x^8$ |
| $E_1$ | $(-x^1, x^2)$ | $(-x^3, x^4)$ | $-x^5$ | $x^6$ | $x^7$ | $x^8$ |

do not take values of 0 and −1, and the coefficient $\eta$ takes values of +1 or −1. Self-normalized subalgebras are denoted by the equality sign. The superscript indicates that in the given subalgebra the parameter value is equal to the superscript (for example, $7.2^0$ designates the subalgebra 7.2 with $\alpha = 0$).

Invariant solutions of rank 1 are constructed on the basis of subalgebras of dimension 2. Among these, subalgebras 2.15 and 2.30 do not satisfy the necessary conditions of existence of an invariant solution. For the remaining subalgebras, invariant solutions are given below. For each submodel, the subalgebra number on whose basis it is constructed, invariants of the corresponding subgroup and the form of the solution are given. To save space, the quote system is not given here; this can be readily obtained by substituting the form of solution into system (1)–(4). If partial or full integrations of the quote system are possible, its result is given. If, among the operators defining the subalgebra there is the operator $X_5$, system (1)–(4) is conveniently written in polar rather than Cartesian coordinates. The Cartesian and polar coordinates are related by

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad u = U\cos\varphi - V\sin\varphi, \quad v = U\sin\varphi - V\cos\varphi.$$

Then, the operators in the polar coordinates take the form

$$X_5 = -\partial_\varphi, \quad X_7 = t\partial_t + r\partial_r - \rho\partial_\rho - p\partial_p, \quad X_8 = r\partial_r + U\partial_U + V\partial_V + 2(\omega - 1)\rho\partial_\rho + 2\omega p\partial_p.$$

The constants of integration are denoted by $c_1$, $\rho_0$, and $p_0$.

TABLE 3

| Subalgebra number | $N$ | Nor$N$ |
|---|---|---|
| 7.1 | $1,2,3,4,5+\alpha 8,6,7+\beta 8$ | $L_8$ |
| 7.2 | $1,2,3,4,5+\alpha 7,6,8$ | $L_8$ |
| 7.3 | $1,2,3,4,5,7,8$ | $=7.3$ |
| 7.4 | $1,2,3,4,6,7,8$ | $L_8$ |
| 6.1 | $1,2,3,4,5+\alpha 7+\beta 8,6$ | $L_8$ |
| 6.2 | $1,2,3,4,5+\alpha 8,7+\beta 8$ | 7.3 |
| 6.3 | $1,2,3,4,5+\delta 7,8$ | 7.3 |
| 6.4 | $1,2,3,4,5+\alpha 8,6+\eta 8$ | $7.2^0$ |
| 6.5 | $1,2,3,4,6,7+\alpha 8$ | $L_8$ |
| 6.6 | $1,2,3,4,5+6,8$ | $7.2^0$ |
| 6.7 | $1,2,3,4,5,8$ | $L_8$ |
| 6.8 | $1,2,5,6,7,8$ | $=6.8$ |
| 6.9 | $1,2,3,6,7,8$ | $=6.9$ |
| 6.10 | $1,2,3,4,7,8$ | $=6.10$ |
| 6.11 | $1,2,3,4,6,8$ | $L_8$ |
| 5.1 | $1,2,3,4,5+\delta 7+\alpha 8$ | 7.3 |
| 5.2 | $1,2,5+\alpha 8,6,7+\beta 8$ | 6.8 |
| 5.3 | $1,2,3,4,5+\alpha 8$ | $L_8$ |
| 5.4 | $1,2,5+\alpha 7,6,8$ | 6.8 |
| 5.5 | $1,2,3,4,5+6+\alpha 8$ | $7.2^0$ |
| 5.6 | $1,2,3,6,7+\alpha 8$ | 7.4 |
| 5.7 | $1,2,3,4,7+\alpha 8$ | 7.3 |
| 5.8 | $3,4,5,7,8$ | $=5.8$ |
| 5.9 | $1,2,5,7,8$ | $=5.9$ |
| 5.10 | $1,3,6,7,8$ | $=5.10$ |
| 5.11 | $1,3,4,7,8$ | 6.10 |
| 5.12 | $1,2,6,7,8$ | 6.7 |
| 5.13 | $1,2,3,7,8$ | $=5.13$ |
| 5.14 | $1,2,3,4+6,7+8$ | $=5.14$ |
| 5.15 | $1,2,3,6,4+7$ | $6.5^0$ |
| 5.16 | $1,2,3,6,8$ | 6.9 |
| 5.17 | $1,2,3,4,8$ | $L_8$ |
| 5.18 | $1,2,3,4,6+\eta 8$ | $7.2^0$ |
| 5.19 | $1,2,3,4,6$ | $L_8$ |
| 4.1 | $1,2,5+\alpha 7+\beta 8,6$ | 6.8 |
| 4.2 | $3,4,5+\alpha 8,7+\beta 8$ | 5.8 |
| 4.3 | $1,2,5+\alpha 8,7+\beta 8$ | 5.9 |
| 4.4 | $3,4,5+\alpha 7,8$ | 5.8 |
| 4.5 | $1,2,5+\alpha 7,8$ | 5.9 |
| 4.6 | $1,3,6,7+\alpha 8$ | 5.10 |
| 4.7 | $1,2,6,7+\alpha 8$ | 6.8 |
| 4.8 | $1,3,4,7+\alpha 8$ | $=4.8$ |
| 4.9 | $1,2,3,7+\alpha 8$ | 5.13 |
| 4.10 | $1,2,5+\alpha 8,6+\eta 8$ | $5.4^0$ |
| 4.11 | $1,2,5+6,8$ | $5.4^0$ |
| 4.12 | $5,6,7,8$ | $=4.12$ |
| 4.13 | $1,6,7,8$ | $=4.13$ |
| 4.14 | $3,4,7,8$ | 5.8 |
| 4.15 | $1+\alpha 2,3,7,8$ | $=4.15$ |
| 4.16 | $2,3,7,8$ | $=4.16$ |
| 4.17 | $1,2,7,8$ | 5.9 |
| 4.18 | $1,3,6,2+7-8$ | $5.6^{-1}$ |

TABLE 3 (Continued)

| Subalgebra number | $N$ | Nor$N$ |
|---|---|---|
| 4.19 | $1,3,4,2+7-8$ | $5.7^{-1}$ |
| 4.20 | $1,3,4+6,7+8$ | $=4.20$ |
| 4.21 | $1,2,3+6,7+8$ | $=4.21$ |
| 4.22 | $1,2,3+7,6$ | $6.5^0$ |
| 4.23 | $1,2,3,4+7$ | 5.7 |
| 4.24 | $1,2,3,6+\eta 8$ | 6.9 |
| 4.25 | $1,3,6,8$ | 5.10 |
| 4.26 | $1,2,6,8$ | 6.8 |
| 4.27 | $1,3,4,8$ | 5.11 |
| 4.28 | $1,2,3,8$ | 6.9 |
| 4.29 | $1,2,3,4+6$ | 6.11 |
| 4.30 | $1,2,3,6$ | 7.4 |
| 4.31 | $1,2,3,4$ | $L_8$ |
| 3.1 | $3,4,5+\alpha 7+\beta 8$ | 5.8 |
| 3.2 | $1,2,5+\delta 7+\alpha 8$ | 5.9 |
| 3.3 | $5+\alpha 8,6,7+\beta 8$ | 4.12 |
| 3.4 | $1,2,5+\alpha 8$ | 6.8 |
| 3.5 | $5+\alpha 7,6,8$ | 4.12 |
| 3.6 | $1,2,5+6+\alpha 8$ | $5.4^0$ |
| 3.7 | $5,7,8$ | $=3.7$ |
| 3.8 | $1,6,7+\delta 8$ | 4.13 |
| 3.9 | $3,4,7+\varepsilon 8$ | 5.8 |
| 3.10 | $1+\alpha 2,3,7+\zeta 8$ | 4.15 |
| 3.11 | $2,3,7+\zeta 8$ | 4.16 |
| 3.12 | $1,2,7+\delta 8$ | 7.3 |
| 3.13 | $6,7,8$ | 4.12 |
| 3.14 | $3,7,8$ | $=3.14$ |
| 3.15 | $1,7,8$ | $=3.15$ |
| 3.16 | $1,6,2+7-8$ | $=3.16$ |
| 3.17 | $3,4,1+7-8$ | 6.10 |
| 3.18 | $1+\alpha 2,3,2+7-8$ | 5.13 |
| 3.19 | $2,3,1+7-8$ | 5.13 |
| 3.20 | $1,3+\alpha 4+6,7+8$ | $=3.20$ |
| 3.21 | $1,4+6,7+8$ | $=3.21$ |
| 3.22 | $3,4,7-8$ | 7.3 |
| 3.23 | $1+\alpha 2,3,7-8$ | 5.13 |
| 3.24 | $2,3,7-8$ | 5.13 |
| 3.25 | $1,6,3+7$ | $4.6^0$ |
| 3.26 | $1,4,3+7$ | $4.8^0$ |
| 3.27 | $1,2,3+7$ | $5.7^0$ |
| 3.28 | $1,3+\alpha 4,4+7$ | $4.8^0$ |
| 3.29 | $1,3+\alpha 4,7$ | 5.11 |
| 3.30 | $1,4,7$ | 5.11 |
| 3.31 | $1,2,7$ | 5.9 |
| 3.32 | $1,6,7$ | 5.10 |
| 3.33 | $1,3,6+\eta 8$ | 4.25 |
| 3.34 | $1,2,6+\eta 8$ | $5.4^0$ |
| 3.35 | $1,6,8$ | 4.13 |
| 3.36 | $3,4,8$ | 5.8 |
| 3.37 | $1+\alpha 2,3,8$ | 4.15 |
| 3.38 | $2,3,8$ | 4.16 |
| 3.39 | $1,2,8$ | 6.8 |
| 3.40 | $1,3,4+6$ | 5.14 |

TABLE 3 (Final)

| Subalgebra number | $N$ | Nor$N$ |
|---|---|---|
| 3.41 | $1, 2, 3+6$ | $6.5^1$ |
| 3.42 | $1, 3, 6$ | 6.9 |
| 3.43 | $1, 2, 6$ | $L_8$ |
| 3.44 | $1, 3, 4$ | 6.10 |
| 3.45 | $1, 2, 3$ | 7.4 |
| 2.1 | $6, 5+\alpha 7+\beta 8$ | 4.12 |
| 2.2 | $5+\alpha 7, 7+\beta 8$ | 3.7 |
| 2.3 | $5+\delta 7, 8$ | 3.7 |
| 2.4 | $5+\alpha 8, 6+\eta 8$ | $3.5^0$ |
| 2.5 | $6, 7+\varepsilon 8$ | 4.12 |
| 2.6 | $3, 7+\delta 8$ | 3.14 |
| 2.7 | $1, 7+\delta 8$ | 3.15 |
| 2.8 | $5+6, 8$ | $3.5^0$ |
| 2.9 | $5, 8$ | 4.12 |
| 2.10 | $7, 8$ | 3.7 |
| 2.11 | $6, 1+7-8$ | 5.12 |
| 2.12 | $3, 2+7-8$ | 5.13 |
| 2.13 | $1, 2+7-8$ | 4.17 |
| 2.14 | $3, 1+\alpha 2+7-8$ | 5.13 |
| 2.15 | $6, 7-8$ | 5.12 |
| 2.16 | $3, 4+7$ | $3.9^0$ |
| 2.17 | $1, 4+7$ | $4.8^0$ |
| 2.18 | $1, 3+\alpha 4+7$ | $4.8^0$ |
| 2.19 | $1, 3+6$ | $5.6^1$ |
| 2.20 | $3, 7$ | 4.14 |
| 2.21 | $1, 7$ | 5.11 |
| 2.22 | $1, 6+\eta 8$ | 4.13 |
| 2.23 | $6, 8$ | 4.12 |
| 2.24 | $3, 8$ | 3.14 |
| 2.25 | $1, 8$ | 4.13 |
| 2.26 | $1, \alpha 3+4+6$ | 5.14 |
| 2.27 | $1, 6$ | 6.9 |
| 2.28 | $3, 4$ | 7.3 |
| 2.29 | $1+\delta 2, 3$ | 6.10 |
| 2.30 | $1, 3$ | 7.4 |
| 2.31 | $2, 3$ | 6.10 |
| 2.32 | $1, 2$ | $L_8$ |
| 1.1 | $5+\delta 7+\alpha 8$ | 3.7 |
| 1.2 | $5+\alpha 8$ | 4.12 |
| 1.3 | $5+6+\alpha 8$ | $3.5^0$ |
| 1.4 | $7+\delta 8$ | 3.7 |
| 1.5 | $1+7-8$ | $3.12^{-1}$ |
| 1.6 | $3+7$ | $3.9^0$ |
| 1.7 | 7 | 5.8 |
| 1.8 | $6+\eta 8$ | $3.5^0$ |
| 1.9 | 8 | 4.12 |
| 1.10 | $3+6$ | 3.41 |
| 1.11 | 6 | 6.8 |
| 1.12 | 3 | 6.10 |
| 1.13 | 1 | 7.4 |

*Submodel* 2.1. The invariants of the group are $r\exp\{(\alpha+\beta)\varphi\}$, $U\exp\{\beta\varphi\}$, $V\exp\{\beta\varphi\}$, $\rho\exp\{(2\beta(\omega-1)-\alpha)\varphi\}$, $p\exp\{(2\beta\omega-\alpha)\varphi\}$. The solution is of the form $U=U_1(\xi)\exp\{-\beta\varphi\}$, $V=V_1(\xi)\exp\{-\beta\varphi\}$, $\rho=\rho_1(\xi)\exp\{(\alpha-2\beta(\omega-1))\varphi\}$, $p=p_1(\xi)\exp\{(\alpha-2\beta\omega)\varphi\}$, $\xi=r\exp\{(\alpha+\beta)\varphi\}$.

*Submodel* 2.2. The invariants of the group are $rt^{-\beta-1}\exp\{-\alpha\beta\varphi\}$, $Ut/r$, $Vt/r$, $\rho r^{2-2\omega}t^{-1+2\omega}$, $pr^{-2\omega}t^{1+2\omega}$. The solution is of the form $U=U_1(\xi)r/t$, $V=V_1(\xi)r/t$, $\rho=\rho_1(\xi)r^{2\omega-2}t^{1-2\omega}$, $p=p_1(\xi)r^{2\omega}t^{-1-2\omega}$, $\xi=rt^{-\beta-1}\exp\{-\alpha\beta\varphi\}$.

*Submodel* 2.3. The invariants of the group are $\delta\varphi+\ln t$, $Ut/r$, $Vt/r$, $\rho r^{2-2\omega}t^{-1+2\omega}$, $pr^{-2\omega}t^{1+2\omega}$. The solution is of the form $U=U_1(\xi)r/t$, $V=V_1(\xi)r/t$, $\rho=\rho_1(\xi)r^{2\omega-2}t^{1-2\omega}$, $p=p_1(\xi)r^{2\omega}t^{-1-2\omega}$, $\xi=\delta\varphi+\ln t$.

*Submodel* 2.4. The invariants of the group are $\alpha\varphi-\eta t+\ln r$, $U/r$, $V/r$, $\rho r^{2-2\omega}$, $pr^{-2\omega}$. The solution is of the form $pr^{-2\omega}$, $U=U_1(\xi)r$, $V=V_1(\xi)r$, $\rho=\rho_1(\xi)r^{2\omega-2}$, $p=p_1(\xi)r^{2\omega}$, $\xi=\alpha\varphi-\eta t+\ln r$.

*Submodel* 2.5. The invariants of the group are $y/x$, $ux^{-\varepsilon/(\varepsilon+1)}$, $vx^{-\varepsilon/(\varepsilon+1)}$, $\rho x^{(1-2\varepsilon(\omega-1))/(\varepsilon+1)}$, $px^{(1-2\varepsilon\omega)/(\varepsilon+1)}$. The solution is of the form $u=u_1(\xi)x^{\varepsilon/(\varepsilon+1)}$, $v=v_1(\xi)x^{\varepsilon/(\varepsilon+1)}$, $\rho=\rho_1(\xi)x^{(2\varepsilon(\omega-1)-1)/(\varepsilon+1)}$, $p=p_1(\xi)x^{(2\varepsilon\omega-1)/(\varepsilon+1)}$, $\xi=y/x$.

*Submodel* 2.6. The invariants of the group are $yt^{-\delta-1}$, $(ut-x)t^{-\delta-1}$, $vt^{-\delta}$, $\rho t^{1-2\delta(\omega-1)}$, $pt^{1-2\delta\omega}$. The solution is of the form $u=u_1(\xi)t^{\delta}+x/t$, $v=v_1(\xi)t^{\delta}$, $\rho=\rho_1(\xi)t^{2\delta(\omega-1)-1}$, $p=p_1(\xi)t^{2\delta\omega-1}$, $\xi=yt^{-\delta-1}$.

*Submodel* 2.7. The invariants of the group are $yt^{-\delta-1}$, $ut^{-\delta}$, $vt^{-\delta}$, $\rho t^{1-2\delta(\omega-1)}$, $pt^{1-2\delta\omega}$. The solution is of the form $u=u_1(\xi)t^{\delta}$, $v=v_1(\xi)t^{\delta}$, $\rho=\rho_1(\xi)t^{2\delta(\omega-1)-1}$, $p=p_1(\xi)t^{2\delta\omega-1}$, $\xi=yt^{-\delta-1}$.

*Submodel* 2.8. The invariants of the group are $t+\varphi$, $U/r$, $V/r$, $\rho r^{2-2\omega}$, $pr^{-2\omega}$. The solution is of the form $U=U_1(\xi)r$, $V=V_1(\xi)r$, $\rho=\rho_1(\xi)r^{2\omega-2}$, $p=p_1(\xi)r^{2\omega}$, $\xi=t+\varphi$.

*Submodel* 2.9. The invariants of the group are $t$, $U/r$, $V/r$, $\rho r^{2-2\omega}$, $pr^{-2\omega}$. The solution is of the form $U=U_1(t)r$, $V=V_1(t)r$, $\rho=\rho_1(t)r^{2\omega-2}$, $p=p_1(t)r^{2\omega}$.

*Submodel* 2.10. The invariants of the group are $y/x$, $ut/x$, $vt/x$, $\rho t^{2\omega-1}x^{2-2\omega}$, $pt^{2\omega+1}x^{-2\omega}$. The solution is of the form $u=u_1(\xi)x/t$, $v=v_1(\xi)x/t$, $\rho=\rho_1(\xi)t^{1-2\omega}x^{2-2\omega}$, $p=p_1(\xi)t^{-1-2\omega}x^{2\omega}$, $\xi=y/x$.

*Submodel* 2.11. The invariants of the group are $y$, $u\exp\{x\}$, $v\exp\{x\}$, $\rho\exp\{(2\omega-1)x\}$, $p\exp\{(2\omega+1)x\}$. The solution is of the form $y$, $u\exp\{x\}$, $v\exp\{x\}$, $\rho\exp\{(2\omega-1)x\}$, $p\exp\{(2\omega+1)x\}$.

*Submodel* 2.12. The invariants of the group are $y-\ln t$, $ut-x$, $vt$, $\rho t^{2\omega-1}$, $pt^{2\omega+1}$. The solution is of the form $u=u_1(\xi)/t+x/t$, $v=v_1(\xi)/t$, $\rho=\rho_1(\xi)t^{1-2\omega}$, $p=p_1(\xi)t^{-1-2\omega}$, $\xi=y-\ln t$.

*Submodel* 2.13. The invariants of the group are $y - \ln t$, $ut$, $vt$, $pt^{2\omega-1}$, $pt^{2\omega+1}$. The solution is of the form $u = u_1(\xi)/t$, $v = v_1(\xi)/t$, $\rho = \rho_1(\xi)t^{1-2\omega}$, $p = p_1(\xi)t^{-1-2\omega}$, $\xi = y - \ln t$.

*Submodel* 2.14. The invariants of the group are $y - \alpha \ln t$, $ut - x + \ln t$, $vt$, $\rho t^{2\omega-1}$, $pt^{2\omega+1}$. The solution is of the form $u = (u_1(\xi) + x - \ln t)/t$, $v = v_1(\xi)/t$, $\rho = \rho_1(\xi)t^{1-2\omega}$, $p = p_1(\xi)t^{-1-2\omega}$, $\xi = y - \alpha \ln t$.

*Submodel* 2.16. The invariants of the group are $y/t - \ln t$, $u - x/t$, $v - \ln t$, $\rho t$, $pt$. The solution is of the form $u = u_1(\xi) + x/t$, $v = v_1(\xi) + \ln t$, $\rho = \rho_1(\xi)/t$, $p = p_1(\xi)/t$, $\xi = y/t - \ln t$.

*Submodel* 2.17. The invariants of the group are $y/t - \ln t$, $u$, $v - \ln t$, $\rho t$, $pt$. The solution is of the form $u = u(\xi)$, $v = v_1(\xi) + \ln t$, $\rho = \rho_1(\xi)/t$, $p = p_1(\xi)/t$, $\xi = y/t - \ln t$.

*Submodel* 2.18. The invariants of the group are $y/t - \alpha \ln t$, $u - \ln t$, $v - \alpha \ln t$, $\rho t$, $pt$. The solution is of the form $u = u_1(\xi) + \ln t$, $v = v_1(\xi) + \alpha \ln t$, $\rho = \rho_1(\xi)/t$, $p = p_1(\xi)/t$, $\xi = y/t - \alpha \ln t$.

*Submodel* 2.19. The invariants of the group are $y$, $u - t$, $v$, $\rho$, $p$. The solution is of the form $u = u_1(y) + t$, $v = v(y)$, $\rho = \rho(y)$, $p = p(y)$.

*Submodel* 2.20. The invariants of the group are $y/t$, $u - x/t$, $v$, $\rho t$, $pt$. The solution is of the form $u = u_1(\xi) + x/t$, $v = v(\xi)$, $\rho = \rho_1(\xi)/t$, $p = p_1(\xi)/t$, $\xi = y/t$.

*Submodel* 2.21. The invariants of the group are $y/t$, $u$, $v$, $\rho t$, $pt$. The solution is of the form $u = u(\xi)$, $v = v(\xi)$, $\rho = \rho_1(\xi)/t$, $p = p_1(\xi)/t$, $\xi = y/t$.

*Submodel* 2.22. The invariants of the group are $y \exp\{-\eta t\}$, $u \exp\{-\eta t\}$, $v \exp\{-\eta t\}$, $\rho \exp\{2(1-\omega)\eta t\}$, $p \exp\{-2\omega\eta t\}$. The solution is of the form $u = u_1(\xi)\exp\{\eta t\}$, $v = v_1(\xi)\exp\{\eta t\}$, $\rho = \rho_1(\xi)\exp\{2(\omega-1)\eta t\}$, $p = p_1(\xi)\exp\{2\omega\eta t\}$, $\xi = y \exp\{-\eta t\}$.

*Submodel* 2.23. The invariants of the group are $y/x$, $u/x$, $v/x$, $\rho x^{2-2\omega}$, $px^{-2\omega}$. The solution is of the form $u = u_1(\xi)x$, $v = v_1(\xi)x$, $\rho = \rho_1(\xi)x^{2\omega-2}$, $p = p_1(\xi)x^{2\omega}$, $\xi = y/x$.

*Submodel* 2.24. The invariants of the group are $t$, $(tu - x)/y$, $v/y$, $\rho y^{2-2\omega}$, $py^{-2\omega}$. The solution is of the form $u = (u_1(t)y + x)/t$, $v = v_1(t)y$, $\rho = \rho_1(t)y^{2\omega-2}$, $p = p_1(t)y^{2\omega}$.

*Submodel* 2.25. The invariants of the group are $t$, $u/y$, $v/y$, $\rho y^{2-2\omega}$, $py^{-2\omega}$. The solution is of the form $u = u_1(t)y$, $v = v_1(t)y$, $\rho = \rho_1(t)y^{2\omega-2}$, $p = p_1(t)y^{2\omega}$.

*Submodel* 2.26. The invariants of the group are $2y - t^2$, $u - \alpha t$, $v - t$, $\rho$, $p$. The solution is of the form $u = u_1(\xi) + \alpha t$, $v = v_1(\xi) + t$, $\rho = \rho(\xi)$, $p = p(\xi)$, $\xi = 2y - t^2$.

*Submodel* 2.27. The invariants of the group are $y$, $u$, $v$, $\rho$, $p$. The solution is of the form $u = u(y)$, $v = v(y)$, $\rho = \rho(y)$, $p = p(y)$.

Integration of the quote system (with accuracy to the transformations defined by the operators $X_3$ and $X_7$) leads to the following two cases:

(1) The solution is restored from the system

$$v = 1/\rho, \qquad \mu u' = u, \qquad 4\mu\rho'/3 = (c_1\rho - p\rho - 1)\rho,$$

$$vp' + \gamma p v' = \frac{\gamma - 1}{R} k_0\Big(\mu\Big(\frac{p}{\rho}\Big)'\Big)' + (\gamma - 1)\mu(u'^2 + 4v'^2/3);$$

(2) The solution is restored from the system

$$v = 0, \qquad p = p_0, \qquad u' = \rho^{\omega}, \qquad \rho' = Ru\rho^{\omega+2}/(k_0 p_0).$$

*Submodel* 2.28. The invariants of the group are $t$, $tu - x$, $tv - y$, $\rho$, $p$. The solution is of the form $u = (u_1(t) + x)/t$, $v = (v_1(t) + y)/t$, $\rho = \rho(t)$, $p = p(t)$.

Integration of the quote system (with accuracy to the transformations defined by the operators $X_1$ and $X_2$) leads to the solution

$$u = x/t, \qquad v = y/t, \qquad \rho = \rho_0/t^2, \qquad p' + 2\gamma p/t = 4(\gamma - 1)\rho_0^{-\omega} t^{\omega-2} p^{\omega}/3.$$

*Submodel* 2.29. The invariants of the group are $t$, $\delta(tu - x) + y$, $v$, $\rho$, $p$. The solution is of the form $u = ((u_1(t) - y)/\delta + x)/t$, $v = v(t)$, $\rho = \rho(t)$, $p = p(t)$.

Integration of the quote system (with accuracy to the transformations defined by the operators $X_2$ and

$X_4$) leads to the solution

$$u = \frac{x}{t} - \frac{y}{\delta t}, \qquad v = 0, \qquad \rho = \frac{\rho_0}{t}, \qquad p' + \frac{\gamma p}{t} = \frac{(\gamma - 1)(4\delta^2/3 + 1)}{\delta^2 t^2}\mu.$$

*Submodel* 2.31. The invariants of the group are $t$, $tu - x$, $v$, $\rho$, $p$. The solution is of the form $u = (u_1(t) + x)/t$, $v = v(t)$, $\rho = \rho(t)$, $p = p(t)$.

Integration of the quote system (with accuracy to the transformations defined by the operators $X_1$ and $X_4$) leads to the solution

$$u = x/t, \qquad v = 0, \qquad \rho = \rho_0/t, \qquad p' + \gamma p/t = 4(\gamma - 1)\rho_0^{-\omega} t^{\omega - 2} p^{\omega}/3.$$

*Submodel* 2.32. The invariants of the group are $t$, $u$, $v$, $\rho$, $p$. The solution is of the form $u = u(t)$, $v = v(t)$, $\rho = \rho(t)$, $p = p(t)$.

Integration of the quote system (with accuracy to the transformations defined by the operators $X_3$ and $X_4$) leads to the state of rest:

$$u = 0, \qquad v = 0, \qquad \rho = \rho_0, \qquad p = p_0.$$

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